UNIVERSITY OF NORTH CAROLINA
Department of Economics

Economics 271 — Dr. Gallant
Midterm Exam — Fall 1998
Oct. 5, 1998

1. (20%) Let X be a random variable that is neither discrete nor continuous.

   (a) Describe how $\mathcal{E}X$ is defined.

   (b) Compute $\mathcal{E}X$ for

   $$X(w) = \begin{cases} \omega & 0 < \omega \le \frac{1}{3} \\ \omega^2 & \frac{1}{3} < \omega \le \frac{2}{3} \\ \omega^3 & \frac{2}{3} < \omega \le 1 \end{cases}$$

   defined on the coin tossing sample space $(\Omega, \mathcal{F}, P)$, where $\Omega = (0, 1]$.

2. (30%) The conditional expectation of Y given X, where Y maps $(\Omega, \mathcal{F})$ into $(\mathcal{Y}, \mathcal{B})$ and X maps $(\Omega, \mathcal{F})$ into $(\mathcal{X}, \mathcal{A})$, is a function $\mathcal{E}(Y|X)(x)$, which maps $(\mathcal{X}, \mathcal{A})$ into $(\mathcal{Y}, \mathcal{B})$, that satisfies the equation

   $$\int_F Y(\omega)\, dP(\omega) = \int_F \mathcal{E}(Y|X)\,[X(\omega)]\; dP(\omega)$$

   for every F of the form $F = X^{-1}(A)$ with $A \in \mathcal{A}$.

   (a) Give the computational formula for $\mathcal{E}(Y|X)(x)$ in these three cases:

      i. $X(\omega) = \sum_{i=0}^{N} x_i I_{F_i}(\omega)$, where $P(F_0) = 0$ and $P(F_i) > 0$ for $i = 1, \ldots, N$.

      ii. X and Y are continuous random variables with density $f_{X,Y}(x, y)$.

      iii. X and Y are discrete random variables with density $f_{X,Y}(x_i, y_j)$.

   (b) Compute $\mathcal{E}(Y|X)(x)$ in the case that X and Y are continuous random variables with density

   $$f_{X,Y}(x, y) = (x + y)\, I_{(0,1]\times(0,1]}(x, y).$$

3. (20%) Let $(\Omega, \mathcal{F}, P)$ be the coin tossing probability space; i.e., $\Omega = (0, 1]$, $\mathcal{F}$ is the smallest σ-algebra that contains all finite unions of sets of the form $(a, b]$, and $P(a, b] = b - a$. Consider the random variable

$$X(\omega) = e^{2\omega},$$

which maps $(\Omega, \mathcal{F}, P)$ into a new probability space $(\mathcal{X}, \mathcal{A}, P_X)$.

(a) What is $\mathcal{X}$ in this new probability space?

(b) What is $\mathcal{A}$ in this new probability space?

(c) If $(c, d] \subset \mathcal{X}$, what is the value of $P_X(c, d]$?

(d) What is the density $f_X(x)$ of the random variable X?

(e) What is the value of $\mathcal{E}X$?

4. (20%) A pair of dice are thrown and the value $\omega = (n_1, n_2)$ is observed where n_1 is the number of spots showing on the first die and n_2 is the number of spots showing on the second.

(a) What is the joint probability density function $f_{X,Y}(x, y)$ of the random variables $X(\omega) = (n_1 + n_2)/2$, $Y(\omega) = n_2/2$?

(b) What is the value of $\mathcal{E}X$?

5. (10%) Show that the intersection of two σ-algebras is a σ-algebra.