UNIVERSITY OF NORTH CAROLINA
Department of Economics

Economics 271 | Dr. Gallant
Midterm Exam | Fall 1997
Oct. 8, 1997

Answer any five questions. Do not turn in answers to more than five questions.

1. (20%) If A and B are subsets of $\mathcal{X}$, and $A_1, A_2, \ldots$ is a sequence of subsets from $\mathcal{X}$, show that the inverse image satisfies these properties:

   (4) $X^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} X^{-1}(A_i)$

   (7) $X^{-1}(\sim A) = \sim X^{-1}(A)$

   You may use these facts without proof in your answer:

   (1) If $A \subset B$, then $X^{-1}(A) \subset X^{-1}(B)$

   (2) $X^{-1}(A \cup B) = X^{-1}(A) \cup X^{-1}(B)$

   (3) $X^{-1}(A \cap B) = X^{-1}(A) \cap X^{-1}(B)$

   (5) $X^{-1}\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} X^{-1}(A_i)$

   (6) If $h(\omega) = g[X(\omega)]$, then $h^{-1}(B) = X^{-1}[g^{-1}(B)]$

2. (20%) Let X be a random variable mapping the measurable space $(\Omega, \mathcal{F})$ onto the measurable space $(\mathcal{X}, \mathcal{A})$. Use the properties stated in Question 1 to show that the collection of sets

$$\mathcal{F}_0 = \left\{F \in \mathcal{F} : F = X^{-1}(A),\ A \in \mathcal{A}\right\}$$

   is a σ-algebra.

3. (20%) Let $(\Omega, \mathcal{F}, P)$ be the coin tossing probability space; i.e., $\Omega = (0,1]$, $\mathcal{F}$ is the smallest σ-algebra that contains all finite unions of sets of the form $(a,b]$, and $P(a,b] = b - a$. Consider the random variable

$$X(\omega) = \frac{1}{3}\omega^3,$$

   which maps $(\Omega, \mathcal{F}, P)$ into a new probability space $(\mathcal{X}, \mathcal{A}, P_X)$.

   What is $\mathcal{X}$ in this new probability space?

What is $\mathcal{A}$ in this new probability space?

If $(c, d] \subset \mathcal{X}$, what is the value of $P_X(c, d]$?

What is the density $f_X(x)$ of the random variable X?

What is the value of $\mathcal{E}X$?

4. (20%) A pair of dice are thrown and the value $\omega = (n_1, n_2)$ is observed where n_1 is the number of spots showing on the first die and n_2 is the number of spots showing on the second. What is the joint probability density function $f_{X,Y}(x, y)$ of the random variables $X(\omega) = n_1 + n_2$, $Y(\omega) = n_2$? What is the value of $\mathcal{E}X$?

5. (20%) Show that the intersection of two σ-algebras is a σ-algebra.

6. (20%) Let the random variable Y be defined by

$$Y = \beta_0 + \beta_1 x + \omega,$$

where ω is gotten by coin tossing as in Question 3 and the value of x is known. Find the density $f_Y(y)$ of the random variable Y. What is the value of $\mathcal{E}Y$?

7. (20%) Suppose that one has a positive valued, finitely additive, set function $P(\cdot)$ defined on an algebra $\mathcal{A}$ of subsets of Ω that assigns $P(\Omega) = 1$ to Ω. How does one extend the definition of $P(\cdot)$ to the smallest σ-algebra $\mathcal{F}$ that contains $\mathcal{A}$? Will $P(\cdot)$ extended to $\mathcal{F}$ be countably additive?