THE PENNSYLVANIA STATE UNIVERSITY
Department of Economics

Economics 501 | Gallant
Midterm Exam | Fall 2013
October 22, 2013

1. (15%) For each of the following, if the statement is true, then prove it, if the statement is false, then give a counter example.

   (a) The collection $\mathcal{A} = \{\emptyset, \Omega, A, \tilde{A}\}$ a σ-algebra.

   (b) The union of two σ-algebras a σ-algebra.

   (c) The intersection of two σ-algebras a σ-algebra.

2. (15%) Let A and B be subsets of $\mathcal{X}$. Show that the inverse image satisfies:

   (a) If $A \subset B$, then $X^{-1}(A) \subset X^{-1}(B)$.

   (b) If $h(\omega) = g[X(\omega)]$, then $h^{-1}(B) = X^{-1}[g^{-1}(B)]$

3. (15%) The conditional expectation of Y given X, where Y maps $(\Omega, \mathcal{F})$ into $(\mathcal{Y}, \mathcal{B})$ and X maps $(\Omega, \mathcal{F})$ into $(\mathcal{X}, \mathcal{A})$, is a function $\mathcal{E}(Y|X)(x)$, which maps $(\mathcal{X}, \mathcal{A})$ into $(\mathcal{Y}, \mathcal{B})$, that satisfies the equation

$$\int_F Y(\omega)\, dP(\omega) = \int_F \mathcal{E}(Y|X)\,[X(\omega)]\, dP(\omega)$$

   for every F of the form $F = X^{-1}(A)$ with $A \in \mathcal{A}$.

   Give the computational formula for $\mathcal{E}(Y|X)(x)$ in these three cases:

   (a) $X(\omega) = \sum_{i=0}^{N} x_i I_{F_i}(\omega)$; x_i distinct, $P(F_0) = 0$ and $P(F_i) > 0$ for $i = 1, \ldots, N$.

   (b) X and Y are continuous random variables with density $f_{X,Y}(x, y)$.

   (c) X and Y are discrete random variables with density $f_{X,Y}(x_i, y_j)$

4. (20%) The coin tossing probability space is $(\Omega, \mathcal{F}, P)$, where $\Omega = (0, 1]$, $\mathcal{F}$ is the smallest σ-algebra containing all intervals of the form $(a, b]$, where $0 \le a \le b \le 1$, and $P(F) = \int I_F(\omega)\, d\omega$. Consider the following events

$$F_1 = (\tfrac{1}{2},\, 1] \qquad \text{heads on the first toss}$$
$$F_2 = (\tfrac{1}{4},\, \tfrac{1}{2}] \cup (\tfrac{3}{4},\, 1] \qquad \text{heads on the second toss}$$
$$F_3 = (\tfrac{1}{8},\, \tfrac{1}{4}] \cup (\tfrac{3}{8},\, \tfrac{1}{2}] \cup (\tfrac{5}{8},\, \tfrac{3}{4}] \cup (\tfrac{7}{8},\, 1] \qquad \text{heads on the third toss}$$

Let $X(\omega) = \frac{1}{3}\omega^3$.

(a) Show that F_1 and F_2 are independent events.

(b) Show that F_1, F_2, and F_3 are independent events.

(c) Derive the density function $f_X(x)$ of X.

(d) Derive the distribution function $F_X(x)$ of X.

5. (15%) Complete the following table

$f(x,y)$	y							
x	1	2	3	4	5	$f(x)$	$F(x)$	$\mathcal{E}(Y\|X)(x)$
1	.02	.02	.03	.05	.04			
2	.05	.04	.04	.04	.05			
3	.02	.01	.03	.06	.03			
4	.03	.03	.06	.05	.05			
5	.05	.04	.05	.06	.05			

6. (20%) Let X and Y be continuous random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute

(a) $f_X(x)$

(b) $\mathcal{E}(X)$

(c) $\mathcal{E}(Y|X)(x)$

(d) $F_{X,Y}(x,y)$