UNIVERSITY OF NORTH CAROLINA
Department of Economics

Economics 271 | Dr. Gallant
Final Exam | Fall 1999
Dec. 10, 1999

1. (15%) Let A be an event from $(\Omega, \mathcal{F}, P)$ that occurs with probability $p = P(A)$ where p is known. Let Y be the random variable on $(\Omega, \mathcal{F}, P)$ defined by $Y(\omega) = I_A(\omega)$.
   (a) Compute $\mathcal{E}Y$.
   (b) Compute $\text{Var}(Y)$.
   (c) Derive the density function $f_Y(y)$ of Y.
   (d) Derive the distribution function $F_Y(y)$ of Y.

2. (10%) Let X_i for $i = 1, \ldots, n$ be independently and identically distributed with mean μ and finite variance σ^2. Estimate $P(\bar{X}_n > 1)$, where $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$, using both Chebishev's inequality and the central limit theorem. You may assume that $1 - \mu > 0$.

3. (10%) Let X_i for $i = 1, \ldots, n$ be independently and identically distributed with common distribution function F_X. Let $g(x)$ be an increasing function with inverse $g^{-1}(y)$. Let $Y_i = g(X_i)$ for $i = 1, \ldots, n$. Prove that the Y_i are independently and identically distributed with common distribution function F_Y and derive F_Y.

4. (5%) Show that $I_{X^{-1}(F)}(\omega) = I_F[X(\omega)]$.

5. (10%) Let X_i be independently and identically distributed with finite variance. Show that $S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ where $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ converges almost surely to $\text{Var}(X)$.

6. (10%) Let (Y_i, X_i) be iid random variables with common density

$$f_{X,Y}(x, y) = n(y|\beta x, 1) f_X(x),$$

where $n(\cdot|\mu, \sigma^2)$ denotes the normal density with mean μ and variance σ^2. Derive the maximum likelihood estimator of β. Note that, unlike the example worked in class, here β and x are scalars, not vectors.

7. (15%) Consider the jointly distributed random variables X and Y with density

$$f(x,y) = \begin{cases} \frac{6}{5}(x^2+y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the marginal density $f(x)$.

(b) Compute the conditional density $f(y|x)$.

(c) Compute $\mathcal{E}(Y|X)(x)$.

8. (5%) Let the random variable X be normally distributied with mean μ and variance σ^2. Compute $\mathcal{E}(e^X)$ and $\mathrm{Var}(e^X)$. Hint: The moment generating function of the normal distribution is $M_X(t) = \exp(\mu t + t^2\sigma^2/2)$.

9. (20%) Let $X_1, \ldots, X_n$ be iid $\mathcal{U}(0,\theta)$. The $\mathcal{U}(0,\theta)$ density is $f_X(x) = \theta^{-1} I_{[0,\theta]}(x)$ and the $\mathcal{U}(0,\theta)$ distribution function is

$$F_X(x) = \begin{cases} 0 & -\infty < x < 0 \\ x/\theta & 0 \le x < \theta \\ 1 & \theta \le x < \infty \end{cases}$$

(a) Show that $P(\max_{1\le i\le n} X_i \le t) = [F_X(t)]^n$.

(b) Compute the mean and variance of $\tilde{\theta}_n = [(n+1)/n] \max_{1\le i\le n} X_i$.

(c) Show that $\tilde{\theta}_n = [(n+1)/n] \max_{1\le i\le n} X_i$ converges in probablility to θ.

(d) Compute the mean and variance of $\hat{\theta}_n = (2/n) \sum_{t=1}^{n} X_i$.

(e) Show that $\hat{\theta}_n = (2/n) \sum_{t=1}^{n} X_i$ converges in probablility to θ.

(f) Which of the two is the better estimator and why.