UNIVERSITY OF NORTH CAROLINA
Department of Economics

Economics 271 Dr. Gallant
Final Exam Fall 1997

1. (10%) Suppose that one has a positive valued, finitely additive, set function $P(\cdot)$ defined on an algebra $\mathcal{A}$ of subsets of Ω that assigns $P(\Omega) = 1$ to Ω. How does one extend the definition of $P(\cdot)$ to the smallest σ-algebra $\mathcal{F}$ that contains $\mathcal{A}$? Will $P(\cdot)$ extended to $\mathcal{F}$ be countably additive?

2. (10%) A pair of dice are thrown and the sum is noted. The throws are repeated until either a sum of 6 or a sum of 7 occurs. What is the probability that the sequence of throws terminates at the 5th roll? What is the expected number of rolls?

3. (10%) Show that the random variables $\mathcal{E}(Y|\mathcal{F}_0)$ and $[Y - \mathcal{E}(Y|\mathcal{F}_0)]$ are orthogonal in the sense that $\mathcal{E}\left\{\mathcal{E}(Y|\mathcal{F}_0)\left[Y - \mathcal{E}(Y|\mathcal{F}_0)\right]\right\} = 0$.

4. (10%) Let $Y = \beta_0 + \beta_1 X + E$ where X and E are independent random variables that are distributed $N(\mu_x, \sigma_x^2)$ and $N(0, \sigma_e^2)$, respectively. Compute $\mathcal{E}(Y)$, $\mathcal{E}(Y|X)$, and find the density of Y.

5. (10%) Find α_0, α_1 that minimize $\text{MSE}(\alpha_0, \alpha_1) = \mathcal{E}\left(Y - \alpha_0 - \alpha_1 X\right)^2$.

6. (15%) Consider the jointly distributed random variables X and Y with density

$$f(x,y) = \begin{cases} \frac{6}{5}(x^2 + y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute the marginal density $f(x)$.

   (b) Compute the conditional density $f(y|x)$.

   (c) Compute $\mathcal{E}(Y|X)(x)$.

   (d) Compute the covariance between X and Y .

   (e) Are X and Y independent?

(f) Compute $P(1/2 \le X \le 1,\, 1/2 \le Y \le 1)$.

7. (10%) Suppose $f_X(x)$ is a density with mean μ and standard deviation σ. Find the density $f_Y(y)$ of the random variable $Y = (X - \mu)/\sigma$. What is the mean and variance of the random variable Y.

8. (10%) Let X_i be independently and identically distributed with finite variance. Show that $S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$, where $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$, converges almost surely to $\text{Var}(X)$.

9. (15%) Consider the random variables

$$Y_i = g(X_i, \theta^o) + E_i, \quad i = 1, \ldots, n,$$

where (X_i, E_i) are independent and indentically distributed, $f_{XE}(x, e) = f_X(x) f_E(e)$, $f_X(x)$ is positive only on the bounded interval (a, b), and $f_E(e)$ is everywhere positive with $\mathcal{E}(E) = 0$ and $0 < \text{Var}(E) < \infty$. The functional form of $g(x, \theta)$ is known. The value of θ^o is unknown but is known to lie in the bounded interval $[c, d]$. Let

$$s_n(\theta) = \frac{1}{n} \sum_{i=1}^n [Y_t - g(X_i, \theta)]^2$$

$$\hat{\theta}_n = \underset{c \le \theta \le d}{\text{argmin}}\, s_n(\theta).$$

Theorem 4.2 implies that

$$\lim_{n \to \infty} \sup_{c \le \theta \le d} |s_n(\theta) - \bar{s}(\theta)| = 0,$$

almost surely, where

$$\bar{s}(\theta) = \text{Var}(E) + \int_a^b [g(x, \theta) - g(x, \theta^o)]^2 f_X(x)\, dx.$$

(a) Assuming that $\bar{s}(\theta)$ has a unique minimum at θ^o over $[c, d]$, show that $\hat{\theta}_n$ converges almost surely to θ^o.

(b) Assuming that

$$\sqrt{n} \frac{d}{d\theta} s_n(\theta^o) = \frac{-2}{\sqrt{n}} \sum_{i=1}^n \frac{d}{d\theta} g(X_i, \theta^o) E_i,$$

use the Central Limit Theorem to show that $\sqrt{n}(d/d\theta) s_n(\theta^o)$ is asymptotically normally distributed. Be sure to include an expression for the variance this asymptotic distribution.