UNIVERSITY OF NORTH CAROLINA
Department of Economics

Economics 271 — Dr. Gallant
Final Exam — Fall 1995

(25%) 1. Let $X_1, \ldots, X_n$ be iid $\mathcal{U}(0,\theta)$. The $\mathcal{U}(0,\theta)$ density is $f_X(x) = \theta^{-1} I_{[0,\theta]}(x)$ and the $\mathcal{U}(0,\theta)$ distribution function is

$$F_X(x) = \begin{cases} 0 & -\infty < x < 0 \\ x/\theta & 0 \le x < \theta \\ 1 & \theta \le x < \infty \end{cases}$$

a. Show that $P(\max_{1\le i\le n} X_i \le t) = [F_X(t)]^n$.

b. Compute the mean and variance of $\tilde{\theta}_n = [(n+1)/n] \max_{1\le i\le n} X_i$.

c. Show that $\tilde{\theta}_n = [(n+1)/n] \max_{1\le i\le n} X_i$ converges in probablility to θ.

d. Compute the mean and variance of $\hat{\theta}_n = (2/n) \sum_{t=1}^{n} X_i$.

e. Show that $\hat{\theta}_n = (2/n) \sum_{t=1}^{n} X_i$ converges in probablility to θ.

f. Which of the two is the better estimator and why.

(15%) 2. Let $X(s)$ be a random variable mapping the sample space $(\mathcal{S}, \mathcal{B}, P)$ onto $(\mathcal{X}, \mathcal{A}, P_X)$.

a. For $B \in \mathcal{B}$ define the image $X(B)$.

b. For $A \in \mathcal{A}$ define the preimage $X^{-1}(A)$.

c. What is the image of $\mathcal{S}$?

d. What is the preimage of $\mathcal{X}$.

e. What is the definition of P_X.

f. What is the definition of the distribution function F_X.

(10%) 3. Let $X(s)$ be a random variable mapping the sample space $(\mathcal{S}, \mathcal{B}, P)$ onto $(\mathcal{X}, \mathcal{A}, P_X)$. Let W=g(X) where $g(x)$ is a one-to-one, increasing function mapping $\mathcal{X}$ onto $\mathcal{W}$ with inverse $x = g^{-1}(w)$. Show that $F_W(t) = F_X[g^{-1}(t)]$.

(15%) 4. Consider the random variable X with density

$$f(x) = \begin{cases} A(1-x^2) & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Compute A.

b. Compute the mean of X.

c. Compute $P(1/2 \le X \le 1)$.

d. Compute the variance of X.

e. Find the density of the random variable $Y = X^3$.

(20%) 5. Consider the jointly distributed random variables X and Y with density

$$f(x,y) = \begin{cases} A(x^2+y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Compute A.

b. Compute the marginal density $f(x)$.

c. Compute the conditional density $f(y|x)$.

d. Compute the covariance between X and Y.

e. Are X and Y independent?

f. Compute $P(1/2 \le X \le 1,\ 1/2 \le Y \le 1)$.

(15%) 6. Let X be a random variable with distribution function $F_X(x)$ and let Y be a random variable with distribution $F_Y(y)$.

a. What is the transformation g(x) such that the random variable W=g(X) has the uniform distribution.

b. What is the transformation g(x) such that the random variable W=g(X) is distributed as the normal.

c. What is the transformation g(x) such that the random variable W=g(X) is distributed as F_Y.