THE PENNSYLVANIA STATE UNIVERSITY
Department of Economics

Economics 501 Gallant
Final Exam Fall 2013

1. (15%) Let $x_1, x_2, \ldots, x_n$ be independent $n(x_i \mid \mu, \sigma^2)$ random variables. Define

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 & \cdots & 0 & 0 \\ & & & & \vdots & & \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{1}{\sqrt{n(n-1)}} & -\frac{n-1}{\sqrt{n(n-1)}} \end{pmatrix}$$

and $z = Ux$.

(a) Show that $z_1 = \sqrt{n}\bar{x}$.

(b) Show that $\sum_{i=2}^{n} z_i^2 = (n-1)s^2$.

(c) Show that the density of z is

$$f(z) = n(z_1 \mid \sqrt{n}\mu,\, \sigma^2) \times \prod_{i=2}^{n} n(z_i \mid 0,\, \sigma^2)$$

(d) Why does 1c imply that $\bar{x}$ and s^2 are independent?

2. (15%) For

$$\hat{\theta}_n = \underset{\theta\in\Theta}{\operatorname{argmin}}\; m_n'(\theta)\tilde{V}^{-1}m_n(\theta),$$

We proved (Section 5.3.2, Generalized Method of Moments) that if

(a) $\lim_{n\to\infty} m_n(\theta) = 0$ a.s. when $\theta = \theta^o$ and $\lim_{n\to\infty} m_n(\theta) \neq 0$ a.s. when $\theta \neq \theta^o$

(b) $\sqrt{n}\, m_n(\theta^o) \xrightarrow{\mathcal{L}} N(0, V)$

(c) There is an estimator $\tilde{V}_n$ of V for which $\lim_{n\to\infty} \tilde{V}_n = V$ a.s.

(d) Some other techical conditions such as domination, closed and bounded Θ, θ^o an interior point of Θ, etc.,

then

$$\sqrt{n}\,(\hat{\theta}_n - \theta^o) \xrightarrow{\mathcal{L}} N\left\{0, \left[M'(\theta^o)V^{-1}M(\theta^o)\right]^{-1}\right\}.$$

If the data $\{x_i\}_{i=1}^n$ are iid with common density $f(x|\theta)$ then

$$m_n(\theta) = \frac{1}{n}\sum_{i=1}^{n}(\partial/\partial\theta)\log f(x|\theta) \tag{1}$$

satisfies the conditions above under the standard regularity conditions for maximum likelihood estimation. For $m_n(\theta)$ chosen as in equation (1), derive V and $M(\theta)$.

3. (10%) Let $X_1, X_2, \ldots, X_n$ be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Show that

$$P\left(\left|\sum_{i=1}^{n} X_i\right| > n\epsilon\right) \leq \sum_{i=1}^{n} P(|X_i| > \epsilon).$$

4. (15%) Let X be a random variable that is neither discrete nor continuous.

(a) Describe how $\mathcal{E}X$ is defined.

(b) Compute $\mathcal{E}X$ for

$$X(w) = \begin{cases} \omega & 0 < \omega \leq \frac{1}{3} \\ \omega^2 & \frac{1}{3} < \omega \leq \frac{2}{3} \\ \omega^3 & \frac{2}{3} < \omega \leq 1 \end{cases}$$

defined on the coin tossing sample space $(\Omega, \mathcal{F}, P)$, where $\Omega = (0, 1]$.

(c) Assuming that $\mathcal{E}|X|^r < \infty$, prove that

$$P(|X| \geq \epsilon) \leq \frac{\mathcal{E}|X|^r}{\epsilon^r}.$$

Do not forget that X is neither discrete nor continuous in your proof.

5. (15%) Let $X_1, \ldots, X_n$ be a random sample from the normal density

$$n(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}.$$

Derive the Wald, Lagrange, and likelihood ratio tests for the hypothesis

$$H : \mu = \mu^o \quad \text{against} \quad A : \mu \neq \mu^o.$$

6. (5%) State the definition of almost sure convergence. State the strong law of large numbers. State the definition of convergence in probability. State the weak law of large numbers.

7. (10%) Show that the random variables $\mathcal{E}(Y|\mathcal{F}_0)$ and $[Y - \mathcal{E}(Y|\mathcal{F}_0)]$ are orthogonal in the sense that $\mathcal{E}\left\{\mathcal{E}(Y|\mathcal{F}_0)\,[Y - \mathcal{E}(Y|\mathcal{F}_0)]\right\} = 0$.

8. (15%) Consider the jointly distributed random variables X and Y with density

$$f(x, y) \quad = \quad \begin{cases} \frac{6}{5}(x^2 + y) & 0 \leq x \leq 1,\, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the marginal density $f(x)$.

(b) Compute the conditional density $f(y|x)$.

(c) Compute $\mathcal{E}(Y|X)(x)$.

(d) Compute the covariance between X and Y .

(e) Are X and Y independent?

(f) Compute $P(1/2 \leq X \leq 1,\, 1/2 \leq Y \leq 1)$.