UNIVERSITY OF NORTH CAROLINA
Department of Economics

Economics 271 — Dr. Gallant
Final Exam — Fall 2002
Dec. 16, 2002

1. (5%) Is the collection

$$\mathcal{A} = \{\emptyset, \Omega, A, \tilde{A}, B, \tilde{B}, A \cup B, A \cup \tilde{B}, \tilde{A} \cup B, \tilde{A} \cup \tilde{B}, A \cap B, A \cap \tilde{B}\}$$

a σ-algebra? If it is, then prove it; if it isn't, then give a example of a set that is missing.

2. (10%) The coin tossing probability space is $(\Omega, \mathcal{F}, P)$ where $\Omega = (0, 1]$, $\mathcal{F}$ is the smallest σ-algebra containing all intervals of the form $(a, b]$, $0 \le a \le b \le 1$, and $P(A) = \int I_A(\omega)\, d\omega$. Give an example of a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ that has $\mathcal{E}X_n = 0$, $\lim_{n\to\infty} \mathrm{Var}(X_n) = \infty$, and $\lim_{n\to\infty} X_n = 0$ a.s.

3. (5%) Suppose that $Y = \mathcal{E}(X|\mathcal{F}_0)$ and that $\mathcal{E}X = \mu$. Show that $\mathcal{E}Y = \mu$.

4. (5%) Suppose that $Y = \mathcal{E}(X|\mathcal{F}_0)$. Show that $Y^2 \le \mathcal{E}(X^2|\mathcal{F}_0)$.

5. (5%) Show that if two events A and B are independent, then so are A and $\tilde{B}$ and $\tilde{A}$ and $\tilde{B}$.

6. (10%) For independently and identically distributed random variables $X_1, X_2, \ldots, X_n$, each with common density $f(x|\theta)$, consider the likelihood

$$\ell(x_1, x_2, \ldots, x_n|\theta) = \prod_{1=1}^{n} f(x_i|\theta)$$

and prior density $p(\theta)$. Set forth the formula for the posterior density. Describe how the posterior density can be computed using the Metropois-Hasting algorithm.

7. (10%) For the density

$$f(x, y) \quad = \quad \begin{cases} \frac{3}{2}(x^2 + y^2) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

derive a rejection algorithm for generating a sample from the density.

8. (15%) Consider the random variable X with density

$$f(x) = \begin{cases} A(16-x^2) & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute A.

(b) Compute the mean of X.

(c) Compute $P(1 \le X \le 3)$.

(d) Compute the variance of X.

(e) Find the density of the random variable $Y = \exp(X)$.

9. (15%) Consider the jointly distributed random variables X and Y with density

$$f(x,y) = \begin{cases} A(x^2+y^2) & 0 \le x \le 1,\, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute A.

(b) Compute the marginal density $f(x)$.

(c) Compute the conditional density $f(y|x)$.

(d) Compute the covariance between X and Y .

(e) Compute $P(0 \le X \le 1/2,\, 0 \le Y \le 1/2)$.

10. (20%) Let $y_i = \beta_0 + \beta_1 x_i + e_i$ where $\{(x_i, e_i)\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables with common mean

$$\mathcal{E}\begin{pmatrix} x_1 \\ e_1 \end{pmatrix} = \begin{pmatrix} \mu_x \\ 0 \end{pmatrix}$$

and common variance

$$\mathcal{E}\begin{pmatrix} (x_1-\mu_x)^2 & (x_1-\mu_x)e_1 \\ e_1(x_1-\mu_x) & e_1^2 \end{pmatrix} = \mathcal{E}\begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{ee} \end{pmatrix}$$

Also, assume that x_i and e_i are independent and have finite fourth moments. Let

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \quad e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

and recall that the least squares estimator is

$$\hat{\beta}_n = (X'X)^{-1}X'y.$$

Below you are asked to verify that several random variables converge in probability. If you would rather work with almost sure convergence instead, you may.

(a) Show that $\hat{\beta}_n = \beta + \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'e\right)$.

(b) Show that $\frac{1}{n}X'X$ converges in probability to $\begin{pmatrix} 1 & \mu_x \\ \mu_x & \sigma_{xx} + \mu_x^2 \end{pmatrix}$

(c) Show that $\det\left(\frac{1}{n}X'X\right)$ converges in probability to σ_{xx}.

(d) Assuming that $\sigma_{xx} > 0$, why do Problems 10b and 10c imply that $\left(\frac{1}{n}X'X\right)^{-1}$ converges in probability to $\frac{1}{\sigma_{xx}}\begin{pmatrix} \sigma_{xx} + \mu_x^2 & -\mu_x \\ -\mu_x & 1 \end{pmatrix}$.

(e) Show that $\frac{1}{n}X'e$ converges in probability to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(f) Use the results above to show that $\hat{\beta}_n$ converges in probability to β.