UNIVERSITY OF NORTH CAROLINA
Department of Economics

Economics 271 — Dr. Gallant
Final Exam — Fall 2001
Dec. 14, 2001

1. (15%) The coin tossing probability space is $(\Omega, \mathcal{F}, P)$ where $\Omega = (0, 1]$, $\mathcal{F}$ is the smallest σ-algebra containing all intervals of the form $(a, b]$, $0 \le a \le b \le 1$, and $P(A) = \int I_A(\omega)\, d\omega$. Let $F_1 = (0,\ 1/2]$, $F_2 = (0,\ 1/4] \cup (1/2,\ 3/4]$ and $X_1(\omega) = I_{F_1}(\omega)$, $X_2(\omega) = I_{F_2}(\omega)$.

   (a) Show that F_1 and F_2 are independent events.

   (b) Derive the densities $f_{X_1}(x_1)$, $f_{X_2}(x_2)$, $f_{X_1,X_2}(x_1, x_2)$, and $f_Y(y)$ of X_1, X_2, (X_1, X_2), and $Y = X_1 + X_2$, respectively.

   (c) Show that $f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$

   (d) F_1 is the event "tails on the first toss", F_2 is the event "tails on the second toss." Write down the event "tails on the third toss" in terms of unions of intervals of the form $(a,\ b]$.

   (e) Is $F_1 \cup F_2$ the event "one tail in the first two tosses" or is it the event "at least one tail on the first two tosses?"

2. (10%) Let $(\Omega, \mathcal{F}, P)$ be the probability space that has $\Omega = (-1,\ 1)$, $\mathcal{F}$ the Borel subsets of Ω, and $P(F) = (1/2) \int_{-1}^{1} I_F(\omega)\, d\omega$. Let $X(\omega) = |\omega|$ map to $(\mathcal{X}, \mathcal{A})$ where $\mathcal{X} = [0, 1)$ and $\mathcal{A}$ is the collection of Borel subsets of $\mathcal{X}$; similarly let $Y(\omega) = \omega$ map to $(\mathcal{Y}, \mathcal{B})$. Let $F = X^{-1}(A)$. Show that if ω is in F then $-\omega$ is in F.

3. (15%) Let the situation be the same as in Question 2. You may use without proof the fact that the result of Question 2 implies that every F in $\mathcal{F}_0 = X^{-1}(\mathcal{A})$ is of the form $F^+ \cup F^-$ where the elements of F^+ are all positive (or zero) and the elements of F^- are the negative of the elements of F^+. Using the definition of conditional expectation, show that $\mathcal{E}(Y|X)(\omega) \equiv 0$. Warning: Do not try to get the answer by deriving the

densities $f_{Y,X}(y,x)$, $f_X(x)$, and $f_{Y|X}(y|x) = f_{Y,X}(y,x)/f_X(x)$ because that approach will not work.

4. (10%) Let $(\Omega, \mathcal{F}, P)$, where $\Omega = (0,1]$, be the coin tossing probability spaces and let X be the random variable

$$X(w) = \begin{cases} \frac{1}{3} & 0 < \omega \leq \frac{1}{3} \\ \omega^2 & \frac{1}{3} < \omega \leq \frac{2}{3} \\ \omega^3 & \frac{2}{3} < \omega \leq 1 \end{cases}$$

   (a) Graph the distribution function of X.

   (b) Compute the expectation of X .

5. (15%) Consider the random variable X with density

$$f(x) = \begin{cases} A(9-x^2) & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute A.

   (b) Compute the mean of X.

   (c) Compute $P(1 \leq X \leq 3)$.

   (d) Compute the variance of X.

   (e) Find the density of the random variable $Y = \exp(X)$.

6. (15%) Consider the jointly distributed random variables X and Y with density

$$f(x,y) = \begin{cases} A(x^2+y) & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute A.

   (b) Compute the marginal density $f(x)$.

   (c) Compute the conditional density $f(y|x)$.

   (d) Compute the covariance between X and Y .

   (e) Compute $P(0 \leq X \leq 1/2,\ 0 \leq Y \leq 1/2)$.

7. (20%) Let $y_i = \beta_0 + \beta_1 x_i + e_i$ where $\{(x_i, e_i)\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables with common mean

$$\mathcal{E}\begin{pmatrix} x_1 \\ e_1 \end{pmatrix} = \begin{pmatrix} \mu_x \\ 0 \end{pmatrix}$$

and common variance

$$\mathcal{E}\begin{pmatrix} (x_1-\mu_x)^2 & (x_1-\mu_x)e_1 \\ e_1(x_1-\mu_x) & e_1^2 \end{pmatrix} = \mathcal{E}\begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{ee} \end{pmatrix}$$

Also, assume that x_i and e_i are independent and have finite fourth moments. Let

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \quad e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

and recall that the least squares estimator is

$$\hat{\beta}_n = (X'X)^{-1}X'y.$$

Below you are asked to verify that several random variables converge in probability. If you would rather work with almost sure convergence instead, you may.

(a) Show that $\hat{\beta}_n = \beta + \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'e\right)$.

(b) Show that $\frac{1}{n}X'X$ converges in probability to $\begin{pmatrix} 1 & \mu_x \\ \mu_x & \sigma_{xx}+\mu_x^2 \end{pmatrix}$

(c) Show that $\det\left(\frac{1}{n}X'X\right)$ converges in probability to σ_{xx}.

(d) Assuming that $\sigma_{xx} > 0$, why do Problems 7b and 7c imply that $\left(\frac{1}{n}X'X\right)^{-1}$ converges in probability to $\frac{1}{\sigma_{xx}}\begin{pmatrix} \sigma_{xx}+\mu_x^2 & -\mu_x \\ -\mu_x & 1 \end{pmatrix}$.

(e) Show that $\frac{1}{n}X'e$ converges in probability to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(f) Use the results above to show that $\hat{\beta}_n$ converges in probability to β.