UNIVERSITY OF NORTH CAROLINA
Department of Economics

Economics 271
Final Exam
Dec. 19, 2000

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1. (10%) Let A and B be disjoint events from $(\Omega, \mathcal{F}, P)$ that occur with probability $p_A = P(A)$ and $p_B = P(B)$, respectively. Let Y be the random variable on $(\Omega, \mathcal{F}, P)$ defined by $Y(\omega) = I_A(\omega) + I_B(\omega)$.

   (a) Compute $\mathcal{E}Y$.

   (b) Compute $\text{Var}(Y)$.

   (c) Derive the density function $f_Y(y)$ of Y.

   (d) Derive the distribution function $F_Y(y)$ of Y.

2. (15%) Consider the random variable X with density

$$f(x) = \begin{cases} A(4 - x^2) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute A.

   (b) Compute the mean of X.

   (c) Compute $P(0 \leq X \leq 1)$.

   (d) Compute the variance of X.

   (e) Find the density of the random variable $Y = \sqrt{X}$.

3. (15%) Consider the jointly distributed random variables X and Y with density

$$f(x, y) = \begin{cases} A(x^2 + y) & 0 \leq x \leq 4,\ 0 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute A.

   (b) Compute the marginal density $f(x)$.

(c) Compute the conditional density $f(y|x)$.

(d) Compute the covariance between X and Y .

(e) Compute $P(1 \leq X \leq 2,\, 1 \leq Y \leq 2)$.

4. (10%) For each of the following, if the statement is true, then prove it, if the statement is false, then give a counter example.

   (a) The collection $\mathcal{A} = \{\emptyset, \Omega, A, \tilde{A}\}$ a σ-algebra.

   (b) The union of two σ-algebras a σ-algebra.

   (c) The intersection of two σ-algebras a σ-algebra.

5. (5%) Suppose that $Y = \mathcal{E}(X|\mathcal{F}_0)$ and that $\mathcal{E}X = \mu$.

   (a) Show that $\mathcal{E}Y = \mu$.

   (b) Show that $Y^2 \leq \mathcal{E}(X^2|\mathcal{F}_0)$.

6. (15%) For each density f_X, support $\mathcal{X}$, and transformation $Y = g(X)$ listed below find the density f_Y and support $\mathcal{Y}$ of the random variable Y. Check your work by verifying that $\int_{\mathcal{Y}} f_Y(y)\, dy = 1$.

   (a) $f_X(x) = 42x^5(1-x),\ \mathcal{X} = \{x : 0 < x < 1\},\ Y = X^3$.

   (b) $f_X(x) = 5e^{-5x},\ \mathcal{X} = \{x : 0 < x < \infty\},\ Y = 2X + 1$.

   (c) $f_X(x) = (2\pi)^{-1/2}e^{-x^2/2};\ \mathcal{X} = \{x : -\infty < x < \infty\},\ Y = 2X + 1$.

   (d) $f_X(x) = (2\pi)^{-1/2}e^{-x^2/2};\ \mathcal{X} = \{x : -\infty < x < \infty\},\ Y = e^X$.

   (e) $f_X(x) = (2\pi)^{-1/2}e^{-x^2/2};\ \mathcal{X} = \{x : -\infty < x < \infty\},\ Y = |X|$.

7. (10%) In a common valuation, oral ascending auction with n bidders, the winner pays the second largest value in a random sample $X_1, \cdots, X_n$ from the common valuation distribution $F_X(x)$. Derive the distribution of the winning bid.
   Hint: If there are three bidders and Y denotes the winning bid then $F_Y(y) = P(X_1 \leq y,\, X_2 \leq y) + P(X_1 \leq y,\, y < X_2) + P(X_2 \leq y,\, y < X_1)$.

8. (20%) Let $y_i = \beta_0 + \beta_1 x_i + e_i$ where $\{(x_i, e_i)\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables with common mean

$$\mathcal{E}\begin{pmatrix} x_1 \\ e_1 \end{pmatrix} = \begin{pmatrix} \mu_x \\ 0 \end{pmatrix}$$

and common variance

$$\mathcal{E}\begin{pmatrix} (x_1-\mu_x)^2 & (x_1-\mu_x)e_1 \\ e_1(x_1-\mu_x) & e_1^2 \end{pmatrix} = \mathcal{E}\begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{ee} \end{pmatrix}$$

Also, assume that x_i and e_i are independent and have finite fourth moments. Let

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \quad e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

and recall that the least squares estimator is

$$\hat{\beta}_n = (X'X)^{-1}X'y.$$

Below you are asked to verify that several random variables converge in probability. If you would rather work with almost sure convergence instead, you may.

(a) Show that $\hat{\beta}_n = \beta + \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'e\right)$.

(b) Show that $\frac{1}{n}X'X$ converges in probability to $\begin{pmatrix} 1 & \mu_x \\ \mu_x & \sigma_{xx}+\mu_x^2 \end{pmatrix}$

(c) Show that $\det\left(\frac{1}{n}X'X\right)$ converges in probability to σ_{xx}.

(d) Assuming that $\sigma_{xx} > 0$, why do Problems 8b and 8c imply that $\left(\frac{1}{n}X'X\right)^{-1}$ converges in probability to $\frac{1}{\sigma_{xx}}\begin{pmatrix} \sigma_{xx}+\mu_x^2 & -\mu_x \\ -\mu_x & 1 \end{pmatrix}$.

(e) Show that $\frac{1}{n}X'e$ converges in probability to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(f) Use the results above to show that $\hat{\beta}_n$ converges in probability to β.