THE PENNSYLVANIA STATE UNIVERSITY
Department of Economics

Economics 501 | Gallant
Homework 12 | Fall 2014
Dec. 2

1. The solution techniques for this problem are the same as those used in the derivation of the asymptotics of the maximum likelihood estimator in Section 5.2.2 of the text.

   Consider the random variables

   $$Y_i = g(X_i, \theta^o) + E_i, \quad i = 1, \dots, n$$

   where (X_i, E_i) are independent and identically distributed, X_i and E_i are independent for each i, the density $f_X(x)$ of X_i is positive on the bounded interval (a, b) and is zero elsewhere, and the density $f_E(e)$ is positive on $(-\infty, \infty)$ with $\mathcal{E}(E) = 0$ and $0 < \mathrm{Var}(E) < \infty$.

   The functional form of $g(x, \theta)$ is known, but the value of θ^o is unknown and must be estimated. It is known that θ^o is in the bounded interval $[c, d]$. Let

   $$s_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} [Y_t - g(X_i, \theta)]^2$$

   and put

   $$\hat{\theta}_n = \underset{c \le \theta \le d}{\mathrm{argmin}}\, s_n(\theta).$$

   Theorem 4.2 implies that

   $$\lim_{n \to \infty} \sup_{c \le \theta \le d} |s_n(\theta) - \bar{s}(\theta)| = 0$$

   almost surely, where

   $$\bar{s}(\theta) = \mathrm{Var}(E) + \int_a^b [g(x, \theta) - g(x, \theta^o)]^2 f_X(x)\, dx.$$

   (a) Compute the expectation of $s_n(\theta)$. Does your computation agree with the value for $\bar{s}(\theta)$ above?

(b) What conditions might you impose to guarantee that $\bar{s}(\theta)$ has a unique minimum at θ^o over $[c, d]$?

(c) If X_i had the uniform distribution on (a, b) and $g(x, \theta) = x\theta$, would $\bar{s}(\theta)$ have a unique minimum at θ^o regardless of what value in $[c, d]$ that θ^o happened to be?

(d) Show that $\hat{\theta}_n$ converges almost surely to θ^o.

(e) Use the first order conditions for the optimization problem min $s_n(\theta)$ and Taylor's theorem to derive the expression

$$\left[\frac{d^2}{d\theta^2} s_n(\bar{\theta})\right] \sqrt{n}(\hat{\theta}_n - \theta^o) = -\sqrt{n}\frac{d}{d\theta} s_n(\theta^o).$$

(f) Show that

$$\sqrt{n}\frac{d}{d\theta} s_n(\theta^o) = \frac{-2}{\sqrt{n}} \sum_{i=1}^{n} \frac{d}{d\theta} g(X_i, \theta^o) E_i.$$

(g) Use the Central Limit Theorem to show that $\sqrt{n}(d/d\theta)s_n(\theta^o)$ is asymptotically normally distributed. Be sure to include an expression for the variance this asymptotic distribution.

(h) Show that $\sqrt{n}(\hat{\theta}_n - \theta^o)$ is asymptotically normally distributed. You may use the fact that

$$\lim_{n\to\infty} \frac{d^2}{d\theta^2} s_n(\bar{\theta}) = \int_c^d \left[\frac{d}{d\theta} g(x, \theta^o)\right]^2 f_X(x)\, dx$$

almost surely without verification. Be sure to include an expression for the variance the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta^o)$.